

Week 4

Recall :

$\Rightarrow$ we only saw this

Theorem : γ optimal in (KP) $\Leftrightarrow$ supp γ is c-cycl. monotone

$$\text{Rmk : } c(x, y) = \begin{cases} \frac{|x-y|^2}{2} \\ -x \cdot y \end{cases} \quad \text{have same minimizers in (KP)}$$

$$\sum_{i=1}^N c(x_i, y_i) \leq \sum_{i=1}^N c(x_{i+1}, y_i) \quad \forall (x_i, y_i)_{i=1, \dots, N} \in \text{supp } \gamma$$

Theorem (Rockafeller) : $(c(x, y) = \frac{|x-y|^2}{2})$

$S \subseteq \mathbb{R}^d \times \mathbb{R}^d$ is c-cyclically monotone $\Leftrightarrow \exists \varphi$ convex : $\mathbb{R}^d \rightarrow \mathbb{R}$ such that $S \subseteq \partial \varphi$

$$\partial \varphi = \bigcup_x \{x\} \times \partial \varphi(x), \quad \partial \varphi(x) := \{y \in \mathbb{R}^d : \forall z \in \mathbb{R}^d, \varphi(z) \geq \varphi(x) + \langle y, z-x \rangle\}.$$

Proof (Rockafeller) :

easy one $\leftarrow$ $\boxed{\Leftarrow}$ Assume $S \subseteq \partial \varphi$, take $\{(x_i, y_i)\}_{i=1, \dots, N} \subset S \subseteq \partial \varphi$, i.e.,

$$\varphi(x) \geq \varphi(x_i) + \langle y_i, x - x_i \rangle \quad \forall x.$$

I want: $\sum_{i=1}^N \langle y_i, x_{i+1} - x_i \rangle \leq 0$. Take $x = x_{i+1}, x_{N+1} = x_1, \sum_i$.

$$\sum_i \varphi(x_{i+1}) \geq \sum_i \varphi(x_i) + \langle y_i, x_{i+1} - x_i \rangle. \quad \checkmark$$

$\Rightarrow$ let $(x_0, y_0) \in S$ fixed. Our φ will need to satisfy, for every

$$\{(x_i, y_i)\}_{i=1, \dots, N} \subseteq S,$$

$$\varphi(x) \geq \overbrace{\langle y_N, x - x_N \rangle}^{(x_N, y_N) \in \partial \varphi} + \varphi(x_N)$$

$$\geq \langle y_N, x - x_N \rangle + \langle y_{N-1}, x_N - x_{N-1} \rangle + \varphi(x_{N-1})$$

$$\geq \dots \geq \langle y_N, x - x_N \rangle + \langle y_{N-1}, x_N - x_{N-1} \rangle + \dots + \langle y_0, x_1 - x_0 \rangle + \underbrace{\varphi(x_0)}$$

$\partial \varphi$ is invariant under $\varphi \mapsto \varphi + c, c \in \mathbb{R}$

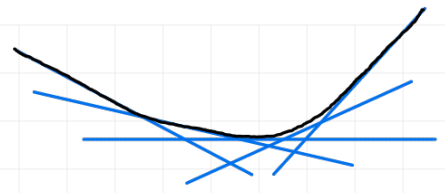
A good definition of φ is :

$$\varphi(x) = \sup \{ \langle y_N, x - x_N \rangle + \dots + \langle y_0, x_1 - x_0 \rangle : N \geq 1, (x_i, y_i) \in S \quad \forall i=1, \dots, N \}$$

fixed

important 1 to N!
(x_0, y_0) fixed

- φ is convex (sup of linear functions)



- Take $(\bar{x}, \bar{y}) \in S$. We claim:

$$\varphi(z) \geq \langle \bar{y}, z - \bar{x} \rangle + \varphi(\bar{x}) \quad \forall z \in \mathbb{R}^d \quad (*)$$

Indeed, let $\alpha < \varphi(\bar{x})$, and by definition, $\exists N \geq 1, (x_1, y_1), \dots, (x_N, y_N)$

such that $\langle y_N, \bar{x} - x_N \rangle + \dots + \langle y_0, x_1 - y_0 \rangle \geq \alpha$.

Consider $(x_i, y_i)_{i=1, \dots, N+1}$ with $x_{N+1} = \bar{x}$, $y_{N+1} = \bar{y}$, admissible in the

$$\begin{aligned} \text{definition of } \varphi: \quad \varphi(z) &\geq \langle \underset{\bar{y}}{y_{N+1}}, z - \underset{\bar{x}}{x_{N+1}} \rangle + \langle y_N, x_{N+1} - x_N \rangle + \dots + \langle y_0, x_1 - x_0 \rangle \\ &\geq \langle \bar{y}, z - \bar{x} \rangle + \alpha \end{aligned}$$

Since α is arbitrarily close to $\varphi(\bar{x})$, we conclude (*).



we have not yet used C -cyclical monotonicity! (We don't know $\varphi \neq +\infty$.)

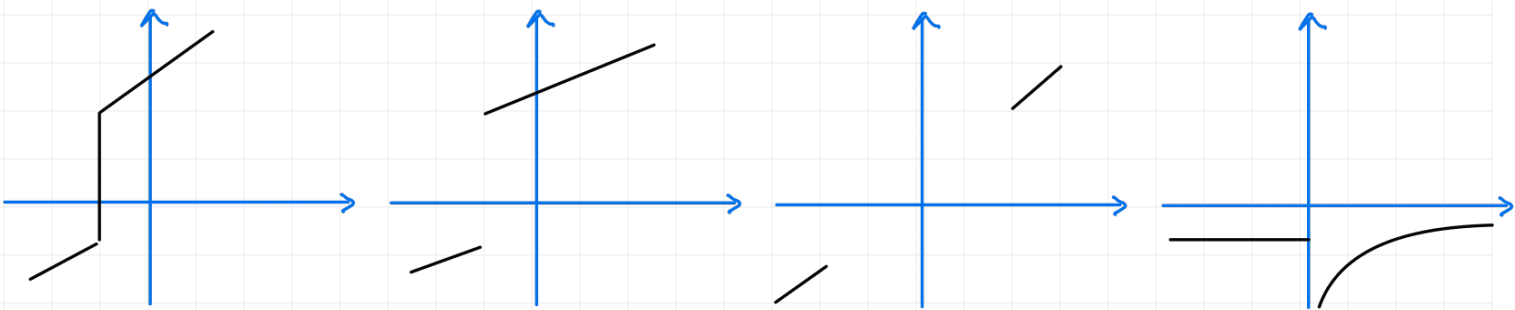
- $\varphi(x_0) \underset{N=1}{\geq} \langle y_1, x_0 - x_1 \rangle + \langle y_0, x_1 - x_0 \rangle \overset{(x_1, y_1) = (x_0, y_0)}{=} 0$

- $\varphi(x_0) \leq 0$, because $\forall (x_i, y_i) \in S, N \geq 1$,
 $\langle y_N, x_0 - x_N \rangle + \dots + \langle y_0, x_1 - x_0 \rangle \leq 0$.

Hence, $\varphi \neq +\infty$.

□

Exercise: What is C -cyclically monotone in $\mathbb{R}^2$?



A tool from convex analysis:

this definition is typically given for convex functions, even if it makes sense for every function

Def: Let $\varphi: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, ($\varphi \not\equiv +\infty$), the Legendre transform of φ is $\varphi^*: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$:

$$\varphi^*(y) := \sup_{x \in \mathbb{R}^d} \{x \cdot y - \varphi(x)\}$$

Convex functions are the supremum of linear functions, $\varphi(x) := \sup_{y \in \mathbb{R}^d} \{x \cdot y + \xi(y)\}$
 $\hookrightarrow$ for every slope y we choose a "height" $\xi(y)$.

Def: Given X, Y metric spaces, $c: X \times Y \rightarrow \mathbb{R}$, $\varphi: X \rightarrow \mathbb{R} \cup \{+\infty\}$, φ is c-convex

if:
$$\varphi(x) = \sup_{y \in \mathbb{R}^d} \{-c(x, y) + \xi(y)\}$$

for some ξ .

! IMPORTANT:
Santambrogio uses a different sign

Def: Let $\varphi: X \rightarrow \mathbb{R} \cup \{+\infty\}$. The c-(Legendre) transform of φ ,

denoted φ^c , is $\varphi^c: Y \rightarrow \mathbb{R} \cup \{+\infty\}$

$$y \mapsto \varphi^c(y) := \sup_{x \in \mathbb{R}^d} \{-c(x, y) - \varphi(x)\}$$

• φ is c-convex $\Leftrightarrow \varphi = \xi^c$ for some ξ .

Exercise (in analogy with Series): $(\varphi^c)^c \leq \varphi$. Equality holds $\Leftrightarrow \varphi$ is c-convex.

Def: Given $\varphi: X \rightarrow \mathbb{R}$ c-convex, the c-subdifferential of φ at $x \in X$ is:

$$\partial^c \varphi(x) := \left\{ y \in Y : \forall z \in X, \varphi(z) \geq \varphi(x) + \underbrace{c(x, y) - c(z, y)}_{\langle y, z-x \rangle \text{ before}} \right\}$$

let
$$\partial^c \varphi := \bigcup_x \{x\} \times \partial^c \varphi(x)$$

Theorem: $S \subseteq X \times Y$ is c-cyclically monotone $\Leftrightarrow \exists \varphi$ c-convex s.t. $S \subseteq \partial^c \varphi$

Proof: literally copy-paste the previous proof:

$\Leftarrow$ Assume $S \subseteq \partial^c \varphi$, take $\{(x_i, y_i)\}_{i=1, \dots, N} \subseteq S \subseteq \partial^c \varphi$,

$\varphi(x) \geq \varphi(x_i) + c(x_i, y_i) - c(x, y_i) \quad \forall x$. I want:

$$\sum_{i=1}^N [c(x_{i+1}, y_i) - c(x_i, y_i)] \geq 0, \text{ take } x = x_{i+1}, x_{i+1} = x_1, \text{ same as before.}$$

⇒ Define:

$$\varphi(x) := \sup \left\{ -c(x, y_N) + c(x_N, y_N) - c(x_{N-1}, y_N) + \dots + c(x_1, y_0) : \right. \\ \left. N \geq 1, (x_i, y_i) \in S \ \forall i = 1, \dots, N \right\}$$

same properties as before.

$$|\varphi(x) - \varphi(y)| \leq |x - y|$$

Remark: If $X = Y$, $c(x, y) = d(x, y)$, φ is c -convex $\Leftrightarrow \varphi$ is 1-Lipschitz.

Indeed, if φ 1-Lipschitz:

$$\varphi(x) \geq \varphi(y) - d(x, y) \Rightarrow \varphi(x) \geq \sup_y \{ \varphi(y) - d(x, y) \} \stackrel{x=y}{\geq} \varphi(x) \\ \Rightarrow \varphi(x) = \sup_y \{ \varphi(y) - d(x, y) \} \Rightarrow \varphi \text{ is } c\text{-convex.}$$

Conversely, if φ is c -convex ($\varphi(x) = \sup_z \{ \xi(z) - d(x, z) \}$) fix $x, y \in X$:

$$-\varphi(x) + \varphi(y) \leq \underbrace{-d(z, y) + \xi(z) + \varepsilon}_{\substack{\downarrow \\ \forall \varepsilon > 0, \exists z: \varphi(y) \leq \xi(z) - d(z, y) + \varepsilon}} \underbrace{-\xi(z) + d(x, z)}_{\substack{\rightarrow \text{Triangle inequality} \\ \varphi(x) \geq \xi(z) - d(x, z)}} \leq d(x, y) + \varepsilon$$

ε is arbitrary $\Rightarrow -\varphi(x) + \varphi(y) \leq d(x, y)$. Changing roles $\Rightarrow \varphi$ 1-lip. $\square$

Prop: Given φ c -convex:

- $\varphi(x) + \varphi^c(y) + c(x, y) \geq 0 \quad \forall x \in X, y \in Y.$
- $\varphi(x) + \varphi^c(y) + c(x, y) = 0 \quad \Leftrightarrow y \in \partial^c \varphi(x).$

Proof: • $\varphi^c(y) \geq -c(x, y) - \varphi(x).$

$$y \in \partial^c \varphi(x) \Leftrightarrow \varphi(z) \geq \varphi(x) + c(x, y) - c(z, y) \quad \forall z \Leftrightarrow$$

$$\Leftrightarrow -\varphi(x) - c(x, y) \geq \sup_z \{ -c(z, y) - \varphi(z) \} = \varphi^c(y)$$

Ⓢ works by first point.

$\square$

Theorem (Kantorovich duality): Assume $c: X \times Y \rightarrow \mathbb{R}$ continuous, bounded below (equivalently, ≥ 0) - If $\inf_{\gamma \in \Gamma(\mu, \nu)} \int c d\gamma = (KP) < +\infty$,

then

$$(KP) = \max_{\varphi, \psi: \varphi(x) + \psi(y) + c(x, y) \geq 0} \int_X -\varphi d\mu + \int_Y -\psi d\nu = (DP)$$

Remark: $c(x, y) = |x - y|^2$ in $\mathbb{R}^d$, $(KP) < +\infty$ as soon as:

$$\int |x|^2 d\mu + \int |y|^2 d\nu < +\infty. \text{ Indeed, } \gamma = \mu \otimes \nu \in \Gamma(\mu, \nu) \text{ has finite cost.}$$

Proof:

(KP) $\geq$ (DP): Take φ, ψ admissible for (DP),

$$\int c(x, y) d\gamma(x, y) \geq \int -\varphi(x) d\gamma + \int -\psi(y) d\gamma = \int -\varphi d\mu + \int -\psi d\nu.$$

Take sup over admissible φ, ψ , and inf over $\gamma \in \Gamma(\mu, \nu) \Rightarrow (KP) \geq (DP)$.

(KP) $\leq$ (DP) Let $\bar{\gamma} \in \Gamma(\mu, \nu)$ optimal $\Rightarrow \text{supp } \bar{\gamma}$ is c -cycl. monotone $\Rightarrow$

by the previous theorem, $\exists \varphi$ c -convex s.t. $\text{supp } \bar{\gamma} \subset \partial^c \varphi$.

By Proposition above $\Rightarrow -\varphi(x) - \varphi^c(y) = c(x, y) \quad \forall (x, y) \in \text{supp } \bar{\gamma}$:

$$\int c(x, y) d\bar{\gamma} = \int -\varphi(x) d\bar{\gamma} + \int -\varphi^c(y) d\bar{\gamma} = \int -\varphi d\mu + \int -\varphi^c d\nu$$

Hence, $(\bar{\gamma}, \varphi, \varphi^c)$ gives equality (KP) and (DP). □

Remark: the optimality of $\bar{\gamma}$ is used only to deduce $\text{supp } \bar{\gamma} \subset \partial^c \varphi$.

Remark: $X = Y$, $c(x, y) = d(x, y)$, φ c -convex $\Leftrightarrow \varphi$ 1-Lipschitz, $\varphi^c = -\varphi$:

$$\min_{\gamma \in \Gamma(\mu, \nu)} \int_{X \times X} d(x, y) d\gamma = \max_{\varphi \text{ 1-Lipschitz}} \int_X \varphi (d\mu - d\nu).$$